ERM Stochastic Analysis Tools - Risk Drivers Revealed

Steven Craighead CERA ASA MAAA

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Abstract

Most stochastic ERM models for life insurance examine only the resultant output (specifically the economic capital), and thereby separates the model results from the key input model assumptions, such as the term structure of interest rates. Now, with ERM modeling, the calculation of economic capital (EC) is very expensive due to the complexity of the products and regulatory controls placed on the industry along with the requirement of a large number of scenarios to produce the empirical distribution of EC. Certain techniques have arisen to reduce this modeling cost, such as grid computing and replicating portfolios, but even with these reductions, a high cost is exacted from the enterprise. However, despite all of the resources dedicated to the generation of EC, the analysis of results is frequently limited to the determination of the empirical distribution and an obligatory examination of the relationships of the five worst and five best scenarios' to the EC.

If we can expand our understanding of the impact of all of the scenarios on the EC, while also targeting specific percentiles of the EC, such as the 98% empirical VaR, our understanding of the enterprise's risk exposure is greatly enhanced. Also, this analysis becomes the springboard for the creation of EC dashboards that allows the analysis of daily changes in the economy on the VaR.

The above is accomplished by use of the quantile regression (QR) modeling of Koenker and Basset [16].

Key Words:

Enterprise Risk Management, Quantile Regression, Economic Capital, Risk Dashboards.

1 Introduction

In the life insurance industry, regulation and/or professional standards require us to conduct computer simulations on different lines of business to determine when the business performs poorly. We model our business as accurately as possible, allowing for interest and asset performance, changing premiums and expense loads. We may or may not make assumptions on the claims count or amount distributions. In addition, we often make many other assumptions such as the term structure of interest rates, future interest rates, projected stock market returns, asset default probabilities, policyholder psychology, and the relationships of our decrements to the level of interest rates or the stock market. Computer simulations reveal the behavior of the business relative to these assumptions. We do not know the actual statistical distribution of our business model results. We assume that the computer simulation results are representative (within some degree of confidence) in certain areas of interest, such as the extreme tail. We need to determine if our models are valid (again within some degree of confidence). If valid, then we calculate either economic capital or stand alone capital within the accuracy of these computer models. In addition, we want to observe the potential risks associated with either the enterprise, product or line of business.

Computer simulations of complex corporate models become very expensive in processing time as the number of scenarios increases. The need to obtain a timely answer often outweighs the need for information from additional scenarios.

In ERM life insurance modeling this cost is reduced by using either predictive modeling, see Craighead [7] or replicating portfolio approaches, see Algorithmics [1].

Most computer business models are limited by the knowledge that we have about the basic assumptions used. We must be careful in how we think about and use these models. At a fundamental level, the models are neither correct nor assumed to be accurate. However, the benefit of using the computer to model actual business products and lines is that we can obtain an understanding of the different risks to which that product or line is exposed. Once we have this understanding, we can consider several methods to reduce the impact of any given risk. Such methods include product redesign, reserve strengthening, deferred expense write downs, asset hedging strategies, stopping rules (rules that recommend when to get out of a market), derivative positions and reinsurance, or the addition of extra capital.

However, once we have gained the basic understanding of the risks and have designed, say, a hedge strategy, we must remember that these models are not accurate, due to oversimplification of the model, lack of knowledge and insight, lack of confidence in the assumptions, or incorrect computer code. We cannot trust the model output as the "truth," but we can trust the knowledge and insight that we have gained from the process of modeling. If done correctly we know both the strengths and weaknesses of the model. For instance, when constructing a hedge to protect against the risks demonstrated by the model, we must not implement a hedge that optimizes against areas of model weakness. Ultimately, the model does not tell us what to do, but the model does make us more comfortable in making business decisions.

It is important to keep a clear perspective when using multiple economic scenarios in computer simulations. We can gain significant insight about the risk exposure from the economy using stochastic simulation. By examining multiple possibilities, we can protect ourselves as best as possible. However, we realize that only one path actually emerges as in the recent economic meltdown. Therefore, the practitioner must continually evaluate the economy and make reasoned business decisions to maintain existing business and to acquire new business.

The risk appetite of company management must also govern these business decisions. Insolvency must be considered and avoided. However, the practitioner cannot remove all risk of insolvency, because the cost of the associated hedges would become so prohibitive that the company could not afford to conduct business. Accordingly, the practitioner should understand where the product or business line places the company at risk and be able to communicate to upper management the specific risk exposure. For a further discussion of the balancing act between company profit and insolvency risk see Craighead [4].

ERM practitioners, valuation actuaries, asset/liability management actuaries, CFOs and CROs of insurance companies confront issues that are vast and complex, including:

1. Calculating the probability and/or impact of bankruptcy either by scenario testing or by determining the company's value at risk.

2. Helping to determine the initial capital allocation for a new line of business.

3. Assuring that reserves are adequate for new and existing lines of business.

4. Understanding how different lines of business are sensitive to the level of interest rates, corporate spreads, volatility of other economic indicators (such as stock indices), and the changes in the levels of these variables.

5. Estimating other risks to which the company is exposed in a timely fashion.

6. Pricing complex policy features to obtain profitability, while maintaining a competitive market position.

7. Aiding in the design and pricing of dynamic hedges to reduce the risk of extreme events.

8. Designing and pricing the securitization of various cashflows to reduce risk based capital requirements.

All of the above issues require timely and accurate valuation of different complex corporate models. When conducting the analysis on models the practitioner goes through the following model life cycle:

1. Collect relevant data.

2. Make relevant assumptions.

3. Construct the model.

4. Validate the model for reasonableness.

5. Revise the model.

After a corporate model is constructed the practitioner uses the results in several ways. Some of these are:

1. Gain insight on the business modeled.

2. Determine risks to the company.

3. Observe the scenarios that give adverse model results.

4. Increase reserves, create hedges or make product enhancements to reduce the risk exposure or adverse results.

The internal company standards and the external regulatory controls require the practitioner to determine risk levels from corporate models. It is of paramount importance to understand the impact that different economic drivers, product designs or investment/disinvestment strategies have on the behavior of a corporate model. This includes the determination of when (and how often) model results from scenarios fall in 'bad' locations. This knowledge allows one to interpret the potential magnitude of the company's risk exposure. While adverse results occur relatively infrequently in scenario testing, the practitioner would like to gain more knowledge of these adverse results without paying the cost of projecting enough scenarios to get the number of "hits" in the region of adverse results needed for statistical validity.

These adverse locations are discovered by first placing a valuation of economic capital on the company's position, scenario by scenario. These valuations are then sorted and put in an increasing or decreasing order. From these ordered results, the location of the adverse results is found at either the highest or lowest valuations. The study and analysis of ordered or sorted samples is done using either order or extreme value statistics or the theory of records. Due to modeling cost, we have a need to approximate the relationship between the input economic scenarios and the EC output results without additional computer processing. Also, if one is able to target the location of adverse results when developing this relationship, all the better.

In Figure 1 we have a non-linear computer corporate model which takes economic scenarios as input and produces certain model output, which represents the EC of the corporate model. Next, we define a risk driver to be a function of the economic scenarios through time that distills the most informative characteristics of the economic scenarios, which have an impact on the model output. The extraction of the time series of the 90-day Treasury bill rate from each scenario is an example of a potential risk driver. Another example is the time series of the spread of the 10-year Treasury note over the 90-day Treasury bill rate.

The dashboard model could be a linear or a nonlinear approximation of the EC at specific percentiles that displays the relationship between the risk drivers and the EC from the original non-linear computer model.

Our dashboard model is based on the use of QR modeling developed by

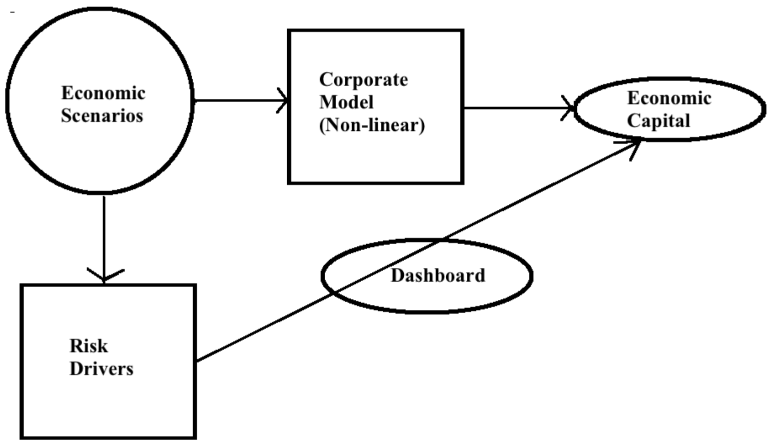

Figure 1: Concept of Risk Drivers

Koenker and Bassett [16]. QR has two major advantages in that it targets specific percentiles (quantiles) and that the model calibration is not influenced by extreme outliers of EC.

See Appendix A for a brief discussion of the theory underpinning Quantile Regression. Also, see Bassett and Koenker [2], Koenker [17], Koenker and Bassett [16], Koenker and Portnoy [18], Portnoy [25], Portnoy and Koenker [26] for further discussions on QR. Buchinsky [3] also has an excellent overview of the theory and applications of QR.

The remainder of this paper takes the following path:

In Section 2 we discuss the illustrative business model that uses the input scenarios and capital results to form the basis of our analysis.

In Section 3 we create a model that displays two risk drivers and the sensitivity of the model results to these drivers.

In Section 4 we discuss how to use a quantile regression model to construct a economic dashboard.

In Section 5 we end the paper with a list of strengths and weaknesses of the QR method and a brief discussion of potential uses.

Finally, Appendix A discusses the QR methodology, Appendix B discusses the generation of the Interest rate scenarios, and Appendix C discusses quantile regression modeling in R.

2 Business Model-Input Economic Scenarios and Economic Capital

For illustrative purposes, we use 10,000 economic scenarios, which are generated from the process outlined in the interest rate Appendix B. This process was one of the first used by Nationwide in the determination of reserve adequacy in the early 1990's. It is a real world process that has arbitrage within the yield curve.

The projection horizon is 20 years with yield curves varying annually. The capital model output is the Equivalent Value of Accumulated Surplus (EVAS).[1] These EVAS values are obtained at the end of the projection period of 20 years and are discounted back to the valuation date. These values are somewhat liberal in that if the company became insolvent in some year prior to year 20, but then recovers subsequently, we do not have knowledge of this event contained in the corresponding twenty-year EVAS value.

The specific business model processed in 1993 is lost to history and the EVAS values have been modified to no longer resemble any of the original values from 1993. However, even though the scenario generation technique as well as the EVAS that were determined from these scenarios are dated, they still supply a rich enough environment to demonstrate the power of quantile regression.

Graphs of the density and S-curve of the capital are in Figure 2 and Figure 3.

The basic statistics on the specific EVAS values are in Table 1.

Variable	**n**	**Min**	$\mathbf{q_1}$	$\widetilde{\mathbf{x}}$	$\bar{\mathbf{x}}$	$\mathbf{q_3}$	**Max**	**s**	**IQR**
LOB	10000	-16.1	19.8	25.1	23.1	28.3	34.9	7.4	8.6

Table 1: LOB Capital Statistics

[1] Equivalent value of accumulated surplus is somewhat similar in concept to a present value, which is scenario dependent. It is also dependent upon the investment strategy used and is obtained by dividing the surplus at the end of the projection period by a growth factor. This factor represents the multiple by which a block of assets would grow from the valuation date to the end of the period in which we are interested. It is computed by accumulating existing assets or an initial lump-sum investment under the interest scenario in question on an after tax basis with the initial investment and any reinvestments being made using the selected investment strategy. The growth factor is the resulting asset amount at the end of the projection period divided by the initial amount at the valuation date, Sedlak [30].

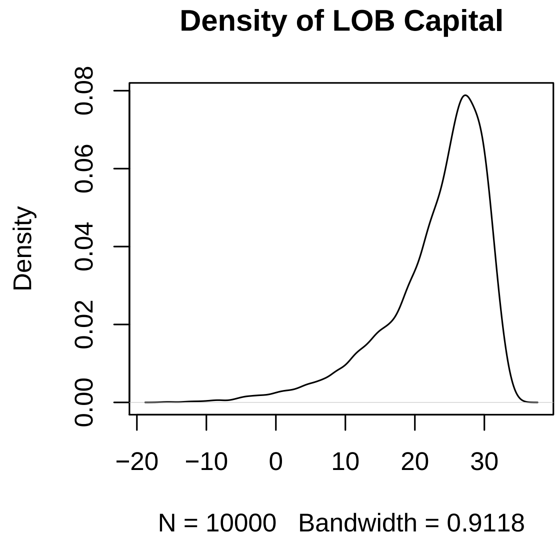

Figure 2: Density of Capital

Since only the interest rate scenarios are available for the data, we restrict our risk drivers to be:

1. The change in the 10-year Treasury bond rates in the input scenarios. These will be denoted Y_t^{10}.
2. The spread between the 10-year Treasury bond rates and the 90-day Treasury bill rates. These will be denoted S_t.

3 Modeling of QR

In the analysis of corporate models, the need to observe the effect of an economic scenario on the model output (specifically economic capital for ERM models) gives the practitioner a critical understanding of the underlying economic risks contained in the model.

Observe the formula

$$R_q = B_{0,q} + B_{1,q}X_1 + B_{2,q}X_2 + \cdots + B_{19,q}X_{19} + B_{20,q}X_{20} + U_q. \quad (1)$$

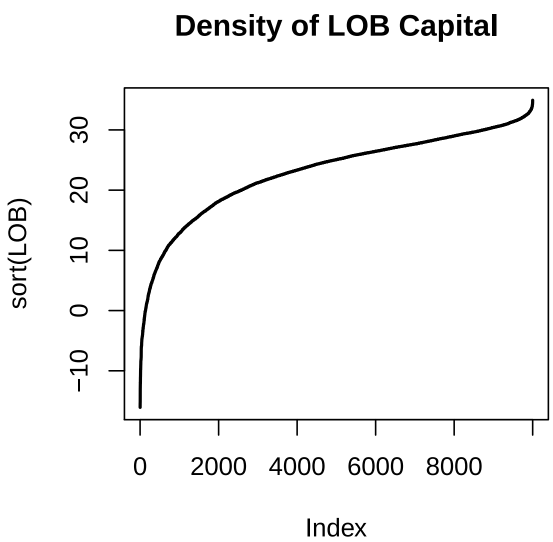

Figure 3: S-Curve of Surplus

R_q is the capital response (specifically at the qth quantile), and the X_t are one of the risk drivers mentioned in Section 2 at the end of each year t. The $B_{t,q}$ are the related coefficients for the specific quantile q and U_q is the error. The assumption that $Quant(U_q) = 0$ leads to the formula of the conditional quantile regression:

$$Quant(R_q) = B_{0,q} + B_{1,q}X_1 + B_{2,q}X_2 + \cdots + B_{19,q}X_{19} + B_{20,q}X_{20}. \quad (2)$$

Koenker and Machado [19] have developed a goodness of fit statistic for QR, which they refer to as the R^1 statistic that corresponds to the LSR R^2 statistic.[2] The R^1 statistic has similar attributes to that of the R^2 statistic

[2]The design matrix of a regression demonstrates its completeness in how well the inner product of the coefficients against the design matrix replicates the responses R_q. In ordinary least squares regression (LSR) the effectiveness (or goodness of fit) is measured by the R^2 statistic. As one adds relevant variables to the design matrix then R^2 will move closer to 1, thus indicating that the design matrix contains sufficient variables. However, if R^2 is close to zero, this implies that variability within the residuals is not well explained with the LSR model. This implies that additional variables should be added to the design matrix or one should try other types of regression. Note: By the use of the Student t test, one can determine if a variable is significant to the model even when R^2 is small.

Time	Coefficient	Standard Error	t value	Pr(> $\|t\|$)	Significant Coefficient	Influence Percent	Ranking
(Intercept)	1.28	0.475	2.696	0.007	NA	NA	NA
1	92.396	32.402	2.852	0.004	92.396	9.5	F
2	-98.147	35.508	-2.764	0.006	-98.147	10.1	E
3	-247.593	36.637	-6.758	0	-247.593	25.5	A
4	-160.356	30.791	-5.208	0	-160.356	16.5	C
5	-180.837	29.105	-6.213	0	-180.837	18.6	B
6	-143.592	37.984	-3.78	0	-143.592	14.8	D
7	-56.589	28.901	-1.958	0.05	0	0	
8	10.903	32.151	0.339	0.735	0	0	
9	-27.137	32.934	-0.824	0.41	0	0	
10	46.794	23.697	1.975	0.048	46.794	4.8	G
11	21.298	33.883	0.629	0.53	0	0	
12	-39.627	26.886	-1.474	0.141	0	0	
13	-26.421	26.396	-1.001	0.317	0	0	
14	4.211	28.431	0.148	0.882	0	0	
15	-17.096	29.455	-0.58	0.562	0	0	
16	10.098	26.652	0.379	0.705	0	0	
17	17.748	27.474	0.646	0.518	0	0	
18	10.361	27.314	0.379	0.704	0	0	
19	-24.687	22.986	-1.074	0.283	0	0	
20	3.466	29.769	0.116	0.907	0	0	
Absolute Sum					969.715		

Table 2: Quantile Regression results for 0.5% 10-year Treasury

for LSR. They also discuss another statistic called a Wald estimator, which can be used in a fashion similar to the Student t statistic to indicate whether a specific variable in the design matrix is significant. (Also see Press et al. [28] for an additional discussion of the Wald estimator.) In Leggett and Craighead [23], they use the R^1 and the Wald estimator as a goodness of fit measure and as a test for variable significance. But, since that time, the theory underpinning QR has improved, and now the use of the Frisch–Newton interior point fitting algorithm returns the measurement of coefficient significance back to the use of Student t statistics as in LSR. Modeling quantile regression models have also been greatly simplified by the use of Koenker's quantile package quantreg [20] in the R language [29]. Koenker has also created several R vignette documents [21] within the quantreg package with several advanced demonstrations to expand the practitioner knowledge as well.

Please refer to Koenker [19, 22] for further discussions of the use and interpretation of these and other statistics.

Our interest for a specific quantile is in its sensitivity to the coefficients

However, low values of R^2 still point to model ineffectiveness. However, an LSR model can still be ineffective with high R^2 due to other problems with the residuals. For instance, if the residuals are serially correlated or if the variance of the residuals is not constant then other problems ensue with the model effectiveness. See Venables and Ripley [34] for a further discussion and for other references relative to the use of LSR.

Time	Coefficient	Standard Error	t value	Pr(> $\|t\|$)	Significant Coefficient	Influence Percent	Ranking
(Intercept)	30.111	0.036	838.224	0	NA	NA	NA
1	316.13	2.311	136.765	0	316.13	19.8	A
2	277.366	1.855	149.504	0	277.366	17.4	B
3	215.893	1.936	111.538	0	215.893	13.6	C
4	162.752	2.492	65.305	0	162.752	10.2	D
5	109.761	2.295	47.833	0	109.761	6.9	E
6	100.18	1.844	54.317	0	100.18	6.3	F
7	62.215	2.329	26.708	0	62.215	3.9	G
8	29.721	1.99	14.932	0	29.721	1.9	
9	8.861	1.783	4.97	0	8.861	0.6	
10	-1.118	2.958	-0.378	0.705	0	0	
11	-21.232	2.985	-7.113	0	-21.232	1.3	
12	-42.542	2.297	-18.517	0	-42.542	2.7	
13	-46.397	3.393	-13.675	0	-46.397	2.9	
14	-53.058	2.219	-23.907	0	-53.058	3.3	
15	-53.757	1.975	-27.223	0	-53.757	3.4	
16	-46.399	2.154	-21.542	0	-46.399	2.9	
17	-17.927	1.745	-10.273	0	-17.927	1.1	
18	-14.432	2.14	-6.743	0	-14.432	0.9	
19	-11.283	1.826	-6.179	0	-11.283	0.7	
20	-3.315	1.607	-2.063	0.039	-3.315	0.2	
Absolute Sum					1593.221		

Table 3: Quantile Regression results for 99.5% 10-year Treasury

Time	Coefficient	Standard Error	t value	Pr(> $\|t\|$)	Significant Coefficient	Influence Percent	Ranking
(Intercept)	4.87	0.838	5.811	0	NA	NA	NA
1	105.298	29.034	3.627	0	105.298	17.5	C
2	64.159	33.175	1.934	0.053	0	0	
3	6.93	27.296	0.254	0.8	0	0	
4	-1.485	28.757	-0.052	0.959	0	0	
5	-1.419	31.594	-0.045	0.964	0	0	
6	-2.549	21.582	-0.118	0.906	0	0	
7	-128.521	25.545	-5.031	0	-128.521	21.4	A
8	-25.51	29.807	-0.856	0.392	0	0	
9	-19.174	27.395	-0.7	0.484	0	0	
10	-3.872	22.871	-0.169	0.866	0	0	
11	-102.406	27.291	-3.752	0	-102.406	17	D
12	62.226	24.814	2.508	0.012	62.226	10.4	F
13	-106.32	28.619	-3.715	0	-106.32	17.7	B
14	-22.804	26.974	-0.845	0.398	0	0	
15	-17.732	24.887	-0.713	0.476	0	0	
16	8.841	25.07	0.353	0.724	0	0	
17	-95.977	23.723	-4.046	0	-95.977	16	E
18	-29.025	24.025	-1.208	0.227	0	0	
19	-3.542	25.964	-0.136	0.891	0	0	
20	5.335	23.544	0.227	0.821	0	0	
Absolute Sum					600.7471		

Table 4: Quantile Regression results for 0.5% Spread

Time	Coefficient	Standard Error	t value	Pr(> $\|t\|$)	Significant Coefficient	Influence Percent	Ranking
(Intercept)	24.829	0.101	245.979	0	NA	NA	NA
1	62.052	2.839	21.859	0	62.052	10.1	B
2	66.058	3.503	18.857	0	66.058	10.8	A
3	57.881	3.517	16.455	0	57.881	9.5	D
4	59.441	3.293	18.052	0	59.441	9.7	C
5	52.954	3.214	16.478	0	52.954	8.6	E
6	45.733	3.537	12.928	0	45.733	7.5	F
7	39.189	3.144	12.464	0	39.189	6.4	G
8	29.201	3.028	9.644	0	29.201	4.8	
9	35.306	3.694	9.557	0	35.306	5.8	
10	33.004	3.799	8.687	0	33.004	5.4	
11	20.313	3.917	5.186	0	20.313	3.3	
12	28.866	3.718	7.763	0	28.866	4.7	
13	10.842	4.127	2.627	0.009	10.842	1.8	
14	2.942	3.245	0.907	0.365	0	0	
15	-9.965	2.532	-3.936	0	-9.965	1.6	
16	-13.632	2.96	-4.605	0	-13.632	2.2	
17	-4.236	4.311	-0.983	0.326	0	0	
18	-8.559	2.827	-3.028	0.002	-8.559	1.4	
19	-16.598	2.937	-5.652	0	-16.598	2.7	
20	-22.794	3.297	-6.914	0	-22.794	3.7	
Absolute Sum					612.38719		

Table 5: Quantile Regression results for 99.5% Spread

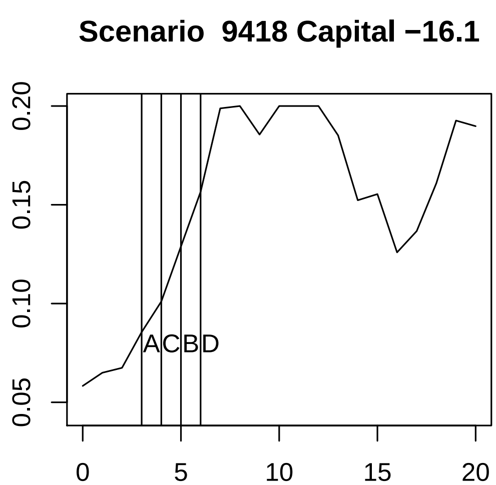

Figure 4: Worst Case Delta 10-year

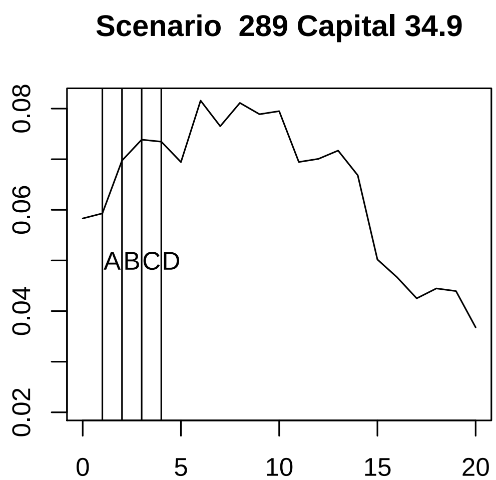

Figure 5: Best Case Delta 10-year

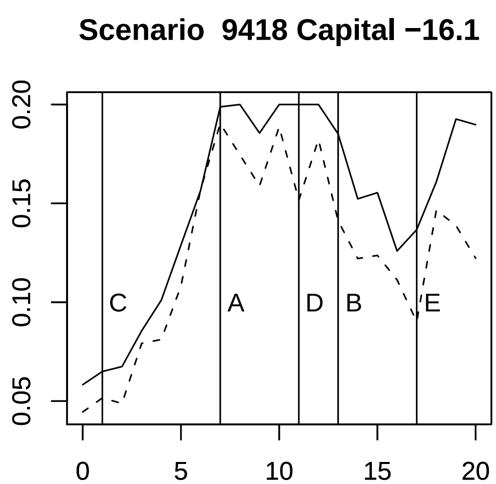

Figure 6: Worst Case Spread

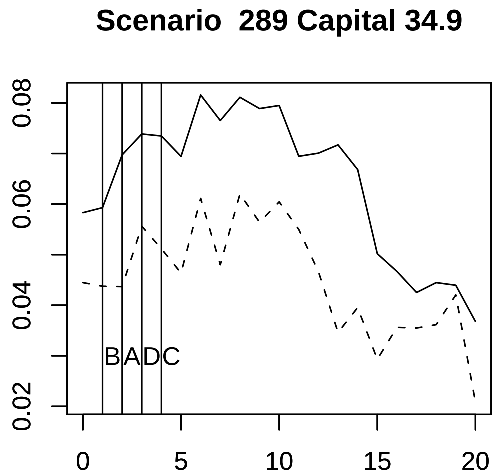

Figure 7: Best Case Spread

through time. By treating the coefficients as a time series, we can observe this effect.

In this section we develop models though methods outlined in Appendix C, which reveals the relevant information that is needed for the practitioner. Initially, the actual value of the QR coefficients is not as critical to our understanding as is their relative magnitude when compared to all of the coefficients. We use the absolute magnitude of the coefficients to locate the year of a specific risk driver as defined in the design matrix of the regression. This approach takes on a qualitative nature in that we do not try to predict the actual percentiles, but we use it to see what influences risk or profit. The pricing actuary can use this qualitative approach to determine design flaws when examining low quantiles and positive upside design features in high quantiles. The valuation actuary can use this type of report to locate various risks and locations of those risks in existing lines of business. This also allows the actuary and the financial engineer to determine risk exposure from embedded options in the business. The financial engineer can also use these methods to improve his or her derivative hedge. This method also examines the effect from all scenarios on the specific R_q. In the past the practitioner may have used different deterministic scenarios to determine the direction of the markets that created risk exposure to the business. However, the deterministic scenarios do not indicate the significance or aid in the determination of the exact location in the projection period that the business is at the highest risk.

Note the following relationship for R_q to the various risk drivers X_t. If value of the X_t can have both positive and negative values, we need only to examine the large $|B_{t,q}|$. Here if one is studying where profit is enhanced at the specific percentage being studied, if X_t is positive and $|B_{t,q}|$ is large and $B_{t,q}$ is positive, R_q increases. If X_t is negative and $B_{t,q}$ is negative, R_q also increases. Just reverse this reasoning if one is interested in determining when the business model is at risk.

In Table 2 we display the QR model regression that correspond to the 0.5% target percentile of the EVAS as modeled against the change in 10-year Treasuries risk driver Y_t^{10} as mentioned in Section 2.

In this table, the values in the coefficient column correspond to the $B_{t,.005}$ in Equation 2. The Standard Error column displays confidence bounds on a specific coefficient's estimate. A good rule of thumb for significance is for the standard error to be less than the absolute value of the associated coefficient. the t–value column displays the value of the Student t statistic which relates the standard error to the coefficient's value. The $Pr(>|t|)$ column is the actual probability associated with the t–value. If $Pr(>|t|)$ is less than 5%, then one can be at least 95% confident in the estimate of the coefficients. These five columns are direct output from the 'rq' function in R as discussed in Appendix C. The additional columns are to display the impact of the coefficients on the model. First, in our analysis we want each risk driver through time to have the same behavior. So, we will exclude the Intercept coefficient in our analysis below, but if a QR model is actually implemented as a dashboard model the Intercept should always be included. The Significant Coefficient column only has non-zero values if the t probability is less than 5%. The absolute sum of the coefficients are at the bottom of this column.

Since we denote the influence of the sign of the $B_{t,q}$ by the risk direction, we are only concerned with the absolute magnitude of the $B_{t,q}$. To determine the influence of specific coefficients, we mimic the process used in Principal Components Analysis (PCA) where the influence of a specific eigenvalue is determined by ranking the ratios of each of the eigenvalues to the sum of all of the absolute values of the eigenvalues. See Dillon [14], Johnson and Wichern [15], and Mardia et al. [24] for a further discussion of PCA. For the

Influence Percent column we use this formula

$$S_t = |B_{t,q}| / \sum_{i=1}^{n} |B_{t,q}|. \tag{3}$$

Note that the effectiveness of this relevance formula only holds if the underlying X_t are of similar magnitude. For instance, this approach would not work if say a risk driver was the combination of a time series of interest rate changes and a time series of changes in equity returns. Since the change in interest rates is less volatile than that of the change in equity returns, larger coefficients would arise from the interest rate changes than from the coefficients associated with the equity changes.

The Ranking column is just an alphabetical ranking to further distinguish which time in the future the risk driver has the greatest impact.

Below, we interpret the different QR models:

1. Change in 10-year Rates – Risk driver Y_t^{10}

    - The 0.5% model corresponds to severe downside possibilities. For instance, year 3 has the most impact on the downside risk, since it has a large negative coefficient and if there is a large positive change in the 10-year rate, the model indicates that things will worsen. From the Influence Percent, we see that this one coefficient explains 25.5% of the change in the model. In addition, note that if the change in the 10-year rates are increasing in years 2 through 6, the company is at increased risk. But, if the change in rates is small or negative in years 1 and 10, then the capital will worsen.

      Let us examine the worst case scenario in the 10,000 set. This is scenario 9418 and it has the worst EVAS value of −16.1. Figure 4 is a graph of the change in 10-year rates through time. It is labeled with the Rankings to the right of the vertical lines. Observe, how the scenario is sharply increasing from 5% to nearly 20% in the first 7 years and that all of the slopes are positive and large in years 3 through 6, with the steepest in year 5 and 6. Notice how our interpretation of the QR model is consistent with these results.

    - Look at the 99.5% upside model in Table 3, notice how the largest significant coefficient starts at year 1 and the significance is high through year 7. So if the slope is sharply increasing in these years, we should see positive increases in the model, which means that the capital will grow.

Out of the 10,000 scenarios, scenario 289 is the best scenario. Notice in Figure 5, where the change in the 10-year rate moderately rises in the beginning years reaching a peak in year seven and then begins to decline in later years. Since the slopes are positive in the early years and negative in the later years the negative significant coefficients in the later years continue to make positive capital.

- Note how in both scenarios, they start with rising rates, but in the worst case, the new money rate is quickly out strips any portfolio returns where the opposite is occurring in the best case, since the portfolio is moderately building up in the first seven years and it continues to contribute in the down years.

2. Spread of 10-year rates over the 90-day rates – Risk driver S_t.

   - In Table 4, the modeled capital suffers if the scenario is strongly positive in years 7, 11, 13, 16, and strongly inverted in year 1 and 12. In Figure 6, even though year 7 is only slightly positive, it is very positive in years 11, 13 and 16 and the spread though positive in years 1 and 12 are very narrow. In fact, we can see that in the worse case scenario, that the yield is very flat from year 1 through year 7 and finally turns more positive in the later years. This reaffirms the fact that the asset portfolio backing the business has very little to no chance to build up in the first seven years, as discussed before.
   - In Table 5, The modeled capital improves if the yield curve is very strongly positive in years 1 through 14, especially in years 1 through 7. Observe in Figure 7, how strongly positive the yield curve is in years 1 through years 15 and the spread narrows after that.
   - Observe how all of the strong spreads throughout most of the projection has built up the initial asset portfolio and this is maintained so that as rates fall the LOB's capital is still adequate.

QR analysis allows the practitioner to conduct risk analysis on several different risk measures. In fact in the past the practitioner did not consider some of the above analyses without extensive additional computer runs. This increased ability may initially raise more questions for the practitioner to analyze, but this type of risk analysis appears to be an excellent tool to conduct these analyses, especially since the model is holistic in that the models are built on the results of all of the scenarios.

4 Dashboard model construction

Here is an outline of turning QR results into dashboards:

1. Pick a specific risk driver based on the scenarios, which can be easily extracted from current daily or weekly economic data.

2. Choose the VaR target percent.

3. Produce the related QR model on the specific risk driver.

4. Use a technique to approximate future values of the economic indicator. For example, if the risk driver is related to the change in 10-year Treasuries, take the current yield curve and produce the implied 10-year forward rates at times where the QR coefficients are significant. Using these forward rates, then replace the QR predictors with the change of rates between the separate 10-year forwards.

To model spreads, create the 90-day forward rates and calculate the spread at each time in the future.

If the risk driver is an equity return there are two approaches to the construction of the dashboard. One, assume that the current economic return is held constant into the future due to a no arbitrage assumption, and all of the predictors in the QR model will be replaced with that single value. Another approach is to actually use a simple economic generator for that equity return and produce multiple equity scenarios and quickly process these future returns through the QR model and average the results.

If the risk driver is either a change in call prices, put prices or equity volatilities, take a similar simulation approach for equity returns.

5 General Comments, Conclusions and Future Research

Koenker and Machado [19], Portnoy [27], and Craighead [5] discuss several ways to display Quantile Regression results.

Other issues surrounding subsampling and data dependency issues involving QR are discussed in Craighead [5].

We will next develop a list of strengths and weaknesses of the QR methodology and finish with a list of further research topics and concluding remarks.

5.1 Strengths and Weaknesses

The strengths of the QR methodology are:

1. The input scenarios tie to the output.
2. The sign and magnitude of the coefficients give insight into risk exposures.
3. Specific percentiles are targets in the output.
4. The model is holistic. The QR results are determined across all scenarios an not just on a small restricted subgroup.
5. The model reveals the influence of a specific period in time to the capital for a specific risk driver.
6. Extreme outliers do not affect the results as much as in LSR.
7. The QR model can be calibrated very fast. The regression on 10,000 scenarios took less than 1 minute in *R*.
8. With the Frisch–Newton methodology very good goodness of fit statistics are finally available.
9. The QR models allows for quick sensitivity testing.
10. Though the examples are linear models, QR model can be done on a nonlinear basis as well, in the R ‘quantreg’ package.

The main weakness of the use of QR are

1. It is relatively complex, but no more so than LSR.
2. Close scrutiny is required to not over simplify the impact of specific risk drivers on the capital models.

5.2 Concluding Remarks

We have data representing the capital on one illustrative line of business, which is modeled against two separate risk driver time series. We apply the QR methodology to this data and develop a QR report and related graphics, which reveal the impact of the two risk driver at specific times. Even though only the best and worse case scenarios compose the graphical analysis, we see that the QR models give another quantitative approach to understand our business.

Also, we discuss how QR models can be used to create dashboard models that allows the monitoring of the change in EC on a more frequent basis.

A The QR Methodology[3]

In multivariate linear regression a column vector of T responses $\{Y_t\}$ are related to a design matrix X of predictors in the following way:

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} + u_t. \quad (4) \\ E[Y_t] &= \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} \quad (5) \end{aligned}$$

Let $t = 1, \cdots, T$ and denote the 'errors' as $\{u_t\}$. These 'errors' are where the predicted value from the formula in X_{ti} does not exactly correspond to the observation Y_t. The $\{u_t\}$ are considered to be independent and identically distributed with a distribution F_u and $E[u_t] = 0$.

Another way to look at the problem is a comparison between a model

$$\hat{Y}_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk}, \quad (6)$$

which tries to predict the responses Y_t from that of some linear combination of the elements of the design matrix. The residuals $u_t = Y_t - \hat{Y}_t$ then are how well or how poorly the model fits the actual responses. In ordinary least

[3] A further presentation of the following material is contained in Buchinsky [3].

squares regression (LSR) the expectation of the residuals are considered to be zero. Also since the expectation operator is linear then $E[u_t] = E[Y_t] - E[\hat{Y}]$.

In multivariate linear regression, the β_i are determined by minimizing the following sum

$$\sum_{1}^{T}(Y_t - (\beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk}))^2. \tag{7}$$

The determination of the $\{\beta_i\}$ is referred to as an LSR or as a l_2 regression estimator. See Portnoy and Koenker [25] for a further discussion of l_2-estimation. After the $\{\beta_i\}$ are determined, the equation relates the sample mean of the Y_t to the predictors.

However, one major difficulty of using LSR is that the values of the $\{\beta_i\}$ can be very sensitive to outliers in the responses $\{Y_t\}$. The area of robust statistics has arisen to deal with this outlier sensitivity. See Venables and Ripley [34] for a series of references on robust statistics.

Koenker and Basset [16] develop quantile regression (QR), where the regression is related to specific quantiles instead of the mean. We will now describe the process.

Let $\mathbf{x_t} = \{1, X_{t1}, X_{t2}, \cdots, X_{tk}\}$, and $\beta_\theta = \{\beta_0, \beta_1, \cdots, \beta_k\}$ and consider the following:

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} + u_{\theta_t} \tag{8}$$

or in matrix format:

$$Y_t = \beta_\theta \mathbf{x_t'} + u_{\theta_t}. \tag{9}$$

$$Quant_\theta(Y_t | \{X_{t1}, X_{t2}, \cdots, X_{tk}\}) = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk} \tag{10}$$

or in matrix format:

$$Quant_\theta(Y_t | \mathbf{x_t}) = \beta_\theta \mathbf{x_t'}. \tag{11}$$

$Quant_\theta(Y_t | \{X_{t1}, X_{t2}, \cdots, X_{tk}\})$ denotes the conditional quantile of Y_t, which is conditional on $\{X_{t1}, X_{t2}, \cdots, X_{tk}\}$, the regression vector. The distribution $F_{u\theta}$ of u_{θ_t}, the error term is not specified. Formula 10 implies that $Quant_\theta(u_{\theta_t} | \{x_{t1}, x_{t2}, \cdots, x_{tk}\}) = 0$ for a specific component vector $\{x_{t1}, x_{t2}, \cdots, x_{tk}\}$.

Let's look at this from the perspective of the residual or error term. Assume that there is a model

$$\hat{Y}_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk}, \tag{12}$$

which tries to predict certain behavior of the responses Y_t that is some linear combination of the elements of the design matrix. The residuals $u_t = Y_t - \hat{Y}_t$ then are a measurement of how well the model relates to the actual responses. The difference between QR and LSR is that instead of the fact that $E[u_t] = 0$ one assumes that $Quant_\theta(u_t) = 0$. This leads to the relationship

$$Quant_\theta(u_t) = Quant_\theta(Y_t - \hat{Y}_t) = 0. \tag{13}$$

The determination of the $\{\beta_i\}$ that allows this relationship to hold will produce the necessary model. However, because the determination of a quantile requires sorting, the quantile operator is not linear. Hence

$$Quant_\theta(Y_t - \hat{Y}_t) \neq Quant_\theta(Y_t) - Quant_\theta(\hat{Y}_t). \tag{14}$$

Koenker and Basset [16] made the following observation: Let Y be a random variable with distribution F. Let $\{y_t : t = 1, \cdots, T\}$ be a random sample on Y. The θth sample quantile for $0 < \theta < 1$ is defined to be any solution of the following minimization problem:

$$min_{b \in \mathbb{R}} \left[\sum_{t \in \{t: y_t \geq b\}} \theta |y_t - b| + \sum_{t \in \{t: y_t < b\}} (1-\theta)|y_t - b| \right]. \tag{15}$$

From the above Koenker and Basset are able to generalize the l_1 regression estimator from the median to all quantiles $0 < \theta < 1$, by finding the $\{\beta_i\}$ that minimizes the following:

$$\sum_{t=1}^{T} \rho_\theta(Y_t - (\beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \cdots + \beta_k X_{tk})), \tag{16}$$

where $0 < \theta < 1$ and

$$\rho_\theta(u) = \begin{cases} \theta u & \text{when } u \geq 0, \\ (\theta - 1)u & \text{when } u < 0. \end{cases} \tag{17}$$

Buchinsky [3] discusses that under certain regularity conditions the consistency and asymptotic normality of $\hat{\beta}_\theta = (\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_k)$ (which is the estimator of $\beta_\theta = (\beta_0, \beta_1, \cdots, \beta_k)$) converges in distribution to a multivariate normal distribution

$$\sqrt{n}((\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_k) - (\beta_0, \beta_1, \cdots, \beta_k)) \quad \xrightarrow{L} \quad N((0, 0, \cdots, 0), \Lambda_\theta). \quad (18)$$

The multivariate normal covariance matrix is

$$\Lambda_\theta = \theta(1-\theta)(E[f_{u\theta}(\mathbf{0}|\mathbf{x_t})\mathbf{x_t}\mathbf{x_t'}])^{-1}E[\mathbf{x_t}\mathbf{x_t'}](E[f_{u\theta}(\mathbf{0}|\mathbf{x_i})\mathbf{x_i}\mathbf{x_i'}])^{-1}. \quad (19)$$

If the density of the error term u_θ at $\mathbf{0}$ is independent of x_t then formula 19 simplifies to

$$\Lambda_\theta = \frac{\theta(1-\theta)E[\mathbf{x_t}\mathbf{x_t'}]^{-1}}{f^2_{u\theta}(0)}, \quad (20)$$

which corresponds to the result in Koenker and Basset [16]. Since the estimator converges in distribution to a multivariate normal distribution, techniques of least squares regression can be used to determine significance of the parameters. For example a Student t statistic can be calculated for each coefficient and from that the retention and/or elimination of a predictor can as easily done as in step-wise regression. This is much simpler than using Wald estimators and the Koenker R^1 factor as in Leggett and Craighead [23].

For any y, if $f_{u\theta}(y|x)$ is independent of x, then the only difference between the quantile regression parameters β_θ for all quantiles θ is in the intercept β_0. In this situation, using quantile regression for risk drivers, would indicate that the risks to a company are symmetric. The same reason that a company profits in a up market is the same reason that it loses in a down market.

The major advantage of using quantile regression is the ability to relate the behavior of a specific quantile of the responses to the design matrix X of predictors. The partial derivative of the conditional quantile of y_t with respect to a specific regressor k is $\partial Quant_\theta(y_t|\mathbf{x_t})/\partial X_{tk}$. Interpret this derivative as the change in the θth conditional quantile due to the change in the kth element of x. Note: when the kth element of x changes this does not imply that when the y_t in a specific quantile θ changes that it will remain in the θth quantile.

B Interest Rate Scenarios

The generation of the yield curves used in the interest rate scenarios is not arbitrage free. This requires setting up a diffusion process of state variables and making sure that the various par bond prices are consistent with the resultant bond pricing partial differential equation. Instead, we used a two-factor model with a lognormal diffusion process on the short-rate (ninety-day) and a lognormal diffusion process on the long-rate (ten-year). This model does not have mean reversion and has fixed boundaries above and below. These fixed boundaries are not reflecting. See Tenney [31, 32, 33] for a further discussion of the behavior and requirements of good interest rate generators.

Below we use the notation Y_t^m, where m denotes the maturity of the interest rate on the yield curve and t denotes the time epoch. The only exception of this notation is that we use Y_t^{90} to denote the value of the ninety-day rate instead of $Y_t^{.25}$. Note $m = \{1, \ldots, 20\}$.

First obtain the initial yield curve. Set Y_0^5 to be Constant Maturity Treasuries (CMT) five-year interest rate for the last day of the year and calculate the ninety-day rate to be

$$Y_0^{90} = Y_0^5 e^{\mu_{90}}. \tag{21}$$

μ_{90} and σ_{90} and σ_{10} are based on a historical lognormal analysis of the short and long rates. We assume below that μ_{10} is zero.

With maturity m, we use the log regression formula

$$N(m) = 1.349ln(2m+1) + 1.051ln(m+1) \tag{22}$$

to assure a "nice" positive or inverted yield curve. This formula precludes the possibility of humped yield curves.

Define the spread slope constant

$$C = (Y_0^5 - Y_0^{90})/N(5). \tag{23}$$

Letting m range from one to twenty, we obtain the entire initial yield curve from

$$Y_0^m = Y_0^{90} + CN(m). \tag{24}$$

For time $t > 0$, the subsequent yield curves are based on a lognormal diffusion processes of the ten-year rate and the ninety-day rate as follows. The ten-year rate is projected as follows:

$$Y_{t+1}^{10} = Y_t^{10} e^{\sigma_{10} Z_{10}}. \tag{25}$$

The ninety-day rate is projected as:

$$Y_{t+1}^{90} = Y_t^{90} e^{\mu_{90} + \sigma_{90} Z_{90}} \tag{26}$$

Z_{90} and Z_{10} are uncorrelated standard normal samples.

These values are then bracketed. The ninety-day brackets are 0.5% and 20% and the brackets of the ten-rates are 1% and 25%.

However, in the belief that inverted yield curves are only observed in a rising interest rate environment, if the yield curve is inverted and the rates are falling (measured by the fact that $Y_{t+1}^{90} > Y_{t+1}^{10}$ and $Y_{t+1}^{10} < Y_t^{10}$) then the Y_{t+1}^{90} is adjusted to be:

$$Y_{t+1}^{90} = Y_{t+1}^{10} e^{\mu_{90}} \tag{27}$$

This new value of Y_{t+1}^{90} is then bracketed as before.

Now define the spread slope constant

$$C = (Y_{t+1}^{10} - Y_{t+1}^{90})/N(10) \tag{28}$$

and obtain the entire yield curve by interpolating by the following formula:

$$Y_{t+1}^{m} = Y_{t+1}^{90} + CN(m) \tag{29}$$

C Quantile Regression Modeling in R

R [29] has become the lingua franca of the statistical world. Though most of the analysis from ERM models occurs in Excel, R is still a good candidate to conduct extensive statistical analyses with the related graphical output. Some of R's benefits are:

1. It is an open source system.

2. It runs on multiple platforms.

3. It is free.

4. It can easily be integrated into multiple packages including Excel.

5. It is constantly improving with cutting edge statistical tools being developed by researchers.

6. World experts such as Koenker have created and continue to maintain high quality packages that can be used by anyone willing to learn a new computer language.

A collection of articles on the use of R has been done by Craighead for the Technology Section newsletter of the Society of Actuaries. To learn the basics of R, here is a list of some recommended articles:

1. Introduction to R [6],
2. Importing Data [8],
3. Manipulating Data [9],
4. Model Formula Framework [10],
5. Functions [11],
6. Graphics [12],
7. Memory Management [13].

Using techniques outlined in the above articles, import the scenario data and the EC data into R, and then manipulate the data so that a data frame contains the EC as the first column of the data frame and the specific risk driver's time series in subsequent columns. Once the data has been prepared in this fashion, access the R 'quantreg' package by Koenker [20] with the 'library' command in R. Next, use the 'rq' function and the Model Formula Framework to create the quantile regression.

Below are some of the commands used in the creation of the QR model for the change in the 10-year Treasury rate study:

```
library(quantreg)
rqcase<-data.frame(cbind(evasadj[,39],nmrs10diff))
rq10<-rq(V1~.,data=rqcase,tau=c(.005,.995),method="fn")
summary(rq10)
```

The first command loads the 'quantreg' package into R.

The second line creates the data frame. The term 'evasadj[,39]' references the EVAS data frame for the 39'th line of business's values and the 'nmrs10diff' data frame is the change in the 10-year Treasuries. These are combined into one data frame using 'cbind' and 'data.frame' commands. The results are stored in rqcase.

The third command line is where the actual QR model is built by the use of the 'rq' function. Using the model formula framework, the first variable in the data frame is named V1 (which is the EVAS) is modeled against all of the other variables in the data frame by the use of the `V1~.` command. The data frame is referenced by the `data =` command and the 0.5% and 99.5% quantiles are input by the `tau=c(.005,.995)` command. The method of fitting indicated by the `method="fn"` command specifies the Frisch–Newton interior point method. Finally the model is stored into the QR object rq10.

The summary command produces these QR results:

```
Call: rq(formula = V1 ~ ., tau = c(0.005, 0.995), data = rqcase, method = "fn")

tau: [1] 0.005

Coefficients:
            Value      Std. Error t value    Pr(>|t|)
(Intercept)    1.28037    0.47487    2.69627    0.00702
t1            92.39614   32.40217    2.85154    0.00436
t2           -98.14702   35.50807   -2.76408    0.00572
t3          -247.59339   36.63717   -6.75798    0.00000
t4          -160.35555   30.79115   -5.20784    0.00000
t5          -180.83712   29.10544   -6.21317    0.00000
t6          -143.59239   37.98391   -3.78035    0.00016
t7           -56.58887   28.90110   -1.95802    0.05026
t8            10.90279   32.15112    0.33911    0.73453
t9           -27.13700   32.93402   -0.82398    0.40997
t10           46.79366   23.69718    1.97465    0.04834
t11           21.29753   33.88260    0.62857    0.52965
t12          -39.62687   26.88594   -1.47389    0.14054
t13          -26.42105   26.39622   -1.00094    0.31688
t14            4.21118   28.43113    0.14812    0.88225
t15          -17.09598   29.45516   -0.58041    0.56165
t16           10.09816   26.65172    0.37889    0.70478
t17           17.74801   27.47426    0.64599    0.51830
t18           10.36056   27.31359    0.37932    0.70446
t19          -24.68656   22.98565   -1.07400    0.28285
t20            3.46574   29.76938    0.11642    0.90732

Call: rq(formula = V1 ~ ., tau = c(0.005, 0.995), data = rqcase, method = "fn")

tau: [1] 0.995

Coefficients:
            Value     Std. Error t value   Pr(>|t|)
(Intercept)  30.11057   0.03592 838.22390   0.00000
t1          316.12966   2.31148 136.76476   0.00000
t2          277.36559   1.85524 149.50425   0.00000
t3          215.89263   1.93559 111.53825   0.00000
t4          162.75161   2.49216  65.30546   0.00000
t5          109.76131   2.29469  47.83283   0.00000
t6          100.17991   1.84436  54.31691   0.00000
t7           62.21496   2.32947  26.70774   0.00000
t8           29.72095   1.99046  14.93167   0.00000
t9            8.86119   1.78286   4.97020   0.00000
t10          -1.11805   2.95772  -0.37801   0.70543
t11         -21.23153   2.98478  -7.11326   0.00000
t12         -42.54223   2.29743 -18.51735   0.00000
t13         -46.39661   3.39269 -13.67545   0.00000
t14         -53.05779   2.21938 -23.90663   0.00000
t15         -53.75742   1.97467 -27.22343   0.00000
t16         -46.39921   2.15389 -21.54210   0.00000
t17         -17.92702   1.74502 -10.27324   0.00000
t18         -14.43172   2.14027  -6.74294   0.00000
t19         -11.28290   1.82601  -6.17899   0.00000
t20          -3.31534   1.60680  -2.06332   0.03911
```

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